Complex and Kähler geometry

Rhys Davies December 23, 2010

These notes describe the basic differential geometry of complex manifolds, and their important specialisation, Kähler manifolds. They begin, however, with an exposition of the relationship between a complex vector space and its underlying real vector space. This lets us highlight the interesting linear algebraic aspects without the complication of fibering them over a manifold.

1 Complex vector spaces

Much of the geometry of manifolds can be described in terms of vector bundles, which associate a vector space to each point of the manifold in a continuous way. Therefore any study of complex manifolds requires some knowledge of complex vector spaces, and we will begin with these. Our focus is on the relationship between a complex vector space and the underlying real vector space, and standard structures on these spaces.

Let V be an n-dimensional complex vector space.[1] Recall that a *Hermitian inner product* on V is a map

$$h \ : \ V \times V \to \mathbb{C}$$

satisfying the following properties:[2]

- $h(v_1, \alpha\, v_2 + v_3) = \alpha\, h(v_1, v_2) + h(v_1, v_3) \ \forall \ \alpha \in \mathbb{C}$ and $v_1, v_2, v_3 \in V$
- $h(v_2, v_1) = \overline{h(v_1, v_2)} \ \forall \ v_1, v_2 \in V$
- $h(v, v) \geq 0 \ \forall \ v \in V$, and $h(v, v) = 0$ if and only if $v = 0$.

Notice that because h is anti-linear in its first argument, it does *not* give an isomorphism of V with its dual space[3] V^*, as a real inner product does. Instead, it gives an isomorphism $V^* \cong \overline{V}$, where $\overline{V}$ is the *complex conjugate* space of V. It is taken to have the same elements as V, with the same additive structure, but scalar multiplication is defined by $\alpha \cdot v := \overline{\alpha} v$. In this language, h is an element of $\overline{V}^{\vee} \otimes V^{\vee}$.

The unitary group $U(V)$ is defined to be the group of linear operators on V which preserve the inner product:

$$U(V) = \{ A \in GL(V) \ \big| \ h(A\, v_1, A\, v_2) = h(v_1, v_2) \ \forall \ v_1, v_2 \in V \} \ .$$

If we choose a basis for V in which h is represented by the identity matrix, then we see that $U(V)$ is isomorphic to $U(n)$, the set of $n \times n$ complex matrices satisfying $A^\dagger A = \mathbf{1}$.

[1] We will restrict ourselves to finite-dimensional spaces throughout.

[2] Sometimes h is taken to be linear in the first argument rather than the second. It is then anti-linear in the other.

[3] The dual space V^* is defined as the space of $\mathbb{C}$-linear maps from V to $\mathbb{C}$, and inherits a vector-space structure from that on $\mathbb{C}$.

1.1 The real perspective

Since the real numbers are a sub-field of the complex numbers, we can also think of V as a $2n$-dimensional real vector space. From this point of view, multiplication by the imaginary unit i is a real-linear operation, which we will denote by J, which squares to minus the identity. To distinguish these perspectives, from now on we will write $V_{\mathbb{C}}$ for the complex vector space, and $V_{\mathbb{R}}$ for the real vector space, and represent multiplication by i, on $V_{\mathbb{R}}$, by the corresponding linear operator J, satisfying $J^2 = -\mathbf{1}$.

What does the Hermitian inner product h on $V_{\mathbb{C}}$ look like from the point of view of $V_{\mathbb{R}}$? Over the real numbers, it makes sense to split h into its real and imaginary parts, which are separately $\mathbb{R}$-linear:

$$\begin{aligned} g(v_1, v_2) &:= \frac{1}{2}\left(h(v_1, v_2) + \overline{h(v_1, v_2)}\right) = \frac{1}{2}\big(h(v_1, v_2) + h(v_2, v_1)\big) \ , \\ \omega(v_1, v_2) &:= -\frac{\mathrm{i}}{2}\left(h(v_1, v_2) - \overline{h(v_1, v_2)}\right) = -\frac{\mathrm{i}}{2}\big(h(v_1, v_2) - h(v_2, v_1)\big) \ . \end{aligned}$$

It is clear that g is symmetric and ω skew-symmetric, and an easy exercise to show that each is non-degenerate, in the sense that $g(v_1, v_2) = 0 \ \forall \ v_2 \iff v_1 = 0$, and similarly for ω. So g is an inner product on $V_{\mathbb{R}}$, and ω a symplectic form.

The inner product g and symplectic form ω are in fact related by the operator J, representing multiplication by i:

$$\omega(v_1, v_2) = g(J\, v_1, v_2) \ .$$

This relation is easy to check from the expressions given above.

Example: $\mathbb{C}^2$

To make the ideas above completely concrete, let us consider $\mathbb{C}^2$ with the standard Hermitian inner product. So let ϵ_1, ϵ_2 be the standard basis for $\mathbb{C}^2$, and define the inner product by

$$h(\epsilon_i, \epsilon_j) = \delta_{i,j} \ .$$

Now we change to the real perspective, $\mathbb{C}^2 \cong \mathbb{R}^4$. We can define a basis over the reals by $e_{i,1} = \epsilon_i \ , \ e_{i,2} = \mathrm{i}\, \epsilon_i$. Then in terms of this basis, we have

$$J\, e_{i,1} = e_{i,2} \ , \quad J\, e_{i,2} = -e_{i,1} \ .$$

So if we take the order of the basis vectors to be $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$, J is represented by the matrix

$$J \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ .$$

In a similar way, we find that the two bilinear forms are represented by the matrices

$$g \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \omega \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

So g and ω are just the standard inner product and symplectic form on $\mathbb{R}^4$. This clearly generalises to any $\mathbb{C}^n$.

We can now consider going the other way, from a real vector space to a complex one. Let V be a real vector space, and J a linear operator on V satisfying $J^2 = -\mathbf{1}$ (this implies that V is even-dimensional, since $(\det J)^2 = \det J^2 = (-1)^{\dim V}$, and $\det J$ is real). We can therefore define an action of the complex numbers of V by $(a + b\,\mathrm{i}) \cdot v := a\,v + b\,J\,v$; it is easy to see that this makes V into a complex vector space. J is called a *complex structure* on V.

An inner product, g, on V is said to be *compatible* with the complex structure if it satisfies $g(J\,v_1, J\,v_2) = g(v_1, v_2)\ \forall\ v_1, v_2 \in V$. In this case we can also construct a symplectic form ω on V, defined by $\omega(v_1, v_2) = g(J\,v_1, v_2)$, and a canonical Hermitian inner product h, given by

$$h(u, v) := g(v_1, v_2) + \mathrm{i}\,\omega(v_1, v_2) .$$

Since $J^2 = -\mathbf{1}$, its eigenvalues are $\pm\mathrm{i}$, so it has no eigenvectors in V. However, we can consider the larger space $\mathbb{C}V := V \otimes_{\mathbb{R}} \mathbb{C}$, and extend J to an operator on this space by the definition

$$J(v \otimes \alpha) = J(v) \otimes \alpha \quad , \quad v \in V\ ,\ \alpha \in \mathbb{C} \tag{1}$$

and $\mathbb{R}$-linearity. It is now possible to diagonalise J, and obtain the decomposition

$$\mathbb{C}V = V^{1,0} \oplus V^{0,1} ,$$

where J acts as multiplication by i on $V^{1,0}$ and multiplication by $-\mathrm{i}$ on $V^{0,1}$. The above decomposition also induces a bi-grading on the exterior algebra of $\mathbb{C}V$; we can define

$$\wedge^{p,q}\mathbb{C}V = \wedge^p V^{1,0} \otimes \wedge^q V^{0,1} .$$

The construction above is actually quite subtle. The complexified space $V \otimes_{\mathbb{R}} \mathbb{C}$ carries two natural complex structures: the first is defined by $\mathrm{i} \cdot (v \otimes \alpha) = v \otimes (\mathrm{i}\alpha)$, and the second is defined by $\mathrm{i} \cdot (v \otimes \alpha) = J(v \otimes \alpha)$, where the right hand side is defined by equation (1). The subspace $V^{1,0}$ is then the subspace on which these two agree. It is naturally isomorphic to the original space V (as a complex vector space), with the explicit isomorphism given by

$$V \ni v \ \mapsto\ (v \otimes 1 - J(v) \otimes \mathrm{i}) \in V^{1,0}$$

We also have naturally that $V^{0,1} \cong \overline{V^{1,0}}$, when each is considered with the 'J' complex structure.

2 Complex manifolds

Since $\mathbb{R}^{2n} \cong \mathbb{C}^n$, any $2n$-dimensional manifold X is locally homeomorphic to $\mathbb{C}^n$, so we obtain nothing interesting from this condition. What we would like is for X to inherit the analytic structure of $\mathbb{C}^n$; we need a consistent notion of a holomorphic function on X.

Let $\{U_\alpha, (x^i_\alpha, y^i_\alpha)\}$ be a covering of X, where we have put the $2n$ real local coordinates into pairs. This lets us define local complex coordinates by $z^i_\alpha = x^i_\alpha + \mathrm{i} y^i_\alpha$, and local holomorphic functions to be complex-valued functions $f = f_1 + \mathrm{i} f_2$ which satisfy the corresponding Cauchy-Riemann equations:

$$\frac{\partial f_1}{\partial x^i_\alpha} = \frac{\partial f_2}{\partial y^i_\alpha} \quad , \quad \frac{\partial f_1}{\partial y^i_\alpha} = -\frac{\partial f_2}{\partial x^i_\alpha} \ .$$

We will call X a complex manifold if on every non-empty intersection $U_\alpha \cap U_\beta$, the restrictions of holomorphic functions from U_α agree with those from U_β. This allows us to consistently refer to 'holomorphic functions' at all points on X. It is easy to check that this condition is equivalent to the transition functions being holomorphic, in the sense that

$$\frac{\partial x^j_\beta}{\partial x^i_\alpha} = \frac{\partial y^j_\beta}{\partial y^i_\alpha} \quad , \quad \frac{\partial x^j_\beta}{\partial y^i_\alpha} = -\frac{\partial y^j_\beta}{\partial x^i_\alpha} \ .$$

These expressions can be written in a much clearer way if we complexify the differential calculus, as is standard in complex analysis. We have the holomorphic and anti-holomorphic derivatives (from now on we drop the reference to the local patch unless required)

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i}\right) \quad , \quad \frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)$$

Then the Cauchy-Riemann equations for a function f are simply $\partial f / \partial \overline{z}^i = 0$.